Some Upper Bounds on Expected Agreement Time of a Probabilistic Local Majority Polling Game *

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Abstract

This paper investigates the expected agreement time of a probabilistic polling game on a connected graph. Given a connected graph G with an assignment of a value in $\{0,1\}$ to each vertex, we consider a polling game on G that repeats the following π-polling forever, where $\pi = (\pi_1, \cdots, \pi_n)$ is a stochastic vector, i.e., $\pi_k \geq 0, \sum_{k=1}^{n} \pi_k = 1$: For k randomly chosen vertices v with probability π_k, synchronously and independently, update their values to $\epsilon \in \{0,1\}$ with probability $N(\epsilon)/|\Gamma(v)|$, where $\Gamma(v)$ is the set of neighbors of v, including v itself, and $N(\epsilon)$ is the number of vertices in $\Gamma(v)$ whose current value is ϵ. Given an initial value assignment, we give some upper bounds on the expected number of π-pollings necessary for the system to reach a global state in which all vertices have the same value, by using a martingale theory. We, in particular, give a good upper bound when G is complete. Note that some special cases are known as Wright-Fisher's and Moran's models in population genetics.

Keywords: *agreement problem, local majority polling, graph theory, Markov chain, martingale.*

1 Introduction

Let $G(V,E)$ be a connected undirected graph with order $|V| = n < \infty$. We assign, to each vertex $v \in V$, a value $\xi(v) \in \{0,1\}$. A *global state* is the set of values that the vertices have and is denoted by $\xi = (\xi(v_1), \cdots, \xi(v_n)) \in \Xi = \{0,1\}^V$. Let $\Gamma(v) = \{v\} \cup \{u \in V : \{u,v\} \in E\}$ be the set of neighbors of v,

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including v itself. The number of vertices having value $\epsilon \in \{0,1\}$ at ξ is denoted by $N_\xi(v,\epsilon)$, i.e., $N_\xi(v,\epsilon) = |\{w \in \Gamma(v) : \xi(w) = \epsilon\}|$ and

$$N_\xi(v,0) + N_\xi(v,1) = |\Gamma(v)| \quad \text{for } v \in V. \tag{1}$$

Let $\pi = (\pi_1, \cdots, \pi_n)$ be a stochastic vector, i.e., $\pi_k \geq 0, \sum_{k=1}^n \pi_k = 1$. This paper discusses a probabilistic polling game on G defined as a repetitive execution of the following probabilistic procedure named *π-polling*: Let $\xi = (\xi(v))_{v \in V} \in \Xi$ be the current global state. Then k randomly chosen vertices $v \in V$ with probability π_k, simultaneously and independently, update their values $\xi(v)$ to $\epsilon \in \{0,1\}$ with probability $N_\xi(v,\epsilon)/|\Gamma(v)|$. Note that the number of ways to choose k vertices from n vertices is $\binom{n}{k}$. Then we see that the $(u_1, u_2, \cdots, u_n)$-polling is the uniformly random polling, where $u_i = \binom{n}{i}/(2^n - 1)$ for $i = 1, \cdots, n$. So that we call $(u_1, \cdots, u_n)$-polling *uniform polling*. The probabilistic polling game defined by δ_n-polling was first introduced by Peleg in connection with distributed agreement and other related problems, where $\delta_k = (\delta_{k,1}, \cdots, \delta_{k,n})$ is the stochastic vector satisfying that $\delta_{k,j} = 1$ if $k = j$; $= 0$ otherwise. We regard the game as an agreement process, where an agreement is achieved when all vertices have the same value [8]. Recently Hassin and Peleg [4] and Nakata et al. [6] independently studied the game. Nakata et al. [6] discussed the game defined by δ_k-polling for $k = 1, \cdots, n$, while Hassin and Peleg [4] concentrated on n-polling. Another slight difference is that in [4], the set of neighbors $\Gamma(v)$ *excludes* v.

The probabilistic polling game defined by π-polling is naturally formulated in terms of Ξ-valued Markov chain $\{X_t\}_{t=0,1,\dots}$, where $X_t = (X_t(v))_{v \in V}$ whose component $X_t(v)$ is the value of v at time t. We consider the probability space $(\Omega, \mathcal{F}, \mathbf{P}_\xi)$ with an initial state $\xi \in \Xi$, i.e., $\mathbf{P}_\xi\{X_0 = \xi\} = 1$. For π-polling, the transition probability from ξ to η is given as follows:

$$p(\xi,\eta) = p(\xi,\xi^A) = \sum_{k=1}^{n} \frac{\pi_k}{\binom{n}{k}} \sum_{U : A \subseteq U \in Sub_k(V)} \prod_{v \in U} \frac{N_\xi(v, \xi^A(v))}{|\Gamma(v)|}, \tag{2}$$

where $Sub_k(V)$ denotes the set of all k-(sub)sets X of V, i.e., $Sub_k(V) = \{X \subseteq V : |X| = k\}$ and

$$\eta = \xi^A(v) = \begin{cases} \xi(v), & \text{if } v \notin A, \\ 1 - \xi(v), & \text{if } v \in A. \end{cases}$$

The following is a partial list of problems concerning this Markov chain:

(I) Except for two trivial absorbing states $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{1} = (1, \cdots, 1)$, are all states $\xi \in \Xi$ transitive?

(II) If the answer for (I) is YES, for a given initial state ξ, calculate the absorbing probability to $\mathbf{0}/\mathbf{1}$, i.e., the probability that all vertices agree on value 0/1.

(III) Estimate the *agreement time* T necessary for the system to reach an absorbing state, where

$$T = \inf\{t \in \mathbf{N} : X_t \in \{\mathbf{0}, \mathbf{1}\}\}. \tag{3}$$

Under our definition of $\Gamma(v)$, i.e., $v \in \Gamma(v)$, the answer for (I) is obviously YES. Under the definition of $\Gamma(v)$ in [4], the answer for (I) is YES unless G is bipartite. Hassin and Peleg, hence, studied non-bipartite graphs in [4].

As for (II), letting $Absorb(\xi, \mathbf{1})$ be the absorbing probability from ξ to $\mathbf{1}$, [4, 6] show that

$$Absorb(\xi, \mathbf{1}) = \sum_{v \in V : \xi(v)=1} |\Gamma(v)| / \sum_{w \in V} |\Gamma(w)|. \tag{4}$$

Now letting $f : \Xi \to \{0, 1, \cdots, 2|E| + |V|\}$, consider the functional of X_t for f, that is

$$x_t = f(X_t) = \sum_{v \in V : X_t(v)=1} |\Gamma(v)|. \tag{5}$$

By [4, 6], we have

$$f(\xi) = \sum_{\eta \in \Xi} p(\xi, \eta) f(\eta). \tag{6}$$

Note that $\{X_t\}_{t=0,1,\cdots}$ is a Markov chain, however $\{x_t\}_{t=0,1,\cdots}$ is *not* Markov in general. Using Eq. (6), we have the next theorem, which plays an essential role in proving Eq. (4).

Theorem 1 ([4, 6]) *Let $\{\mathcal{F}_t\}_{t=0,1,\cdots}$ be the filtration of Markov chain $\{X_t\}_{t=0,1,\cdots}$. Then $(x_t, \mathcal{F}_t)$ is a martingale. That is for any t*

$$\mathbf{E}_\xi[x_{t+1} | \mathcal{F}_t] = x_t \quad \mathbf{P}_\xi\text{-a.s.},$$

where $\mathbf{E}_\xi[\cdot]$ is the expectation with initial state ξ.

By virtue of Theorem 1, since $T = \inf\{t : X_t^{(k)} \in \{\mathbf{0}, \mathbf{1}\}\}$ has the stopping time property, we apply, to x_t, the optional stopping theorem [5, Corollary 3-16] to show

$$Absorb(\xi, \mathbf{1}) \cdot \sum_{v \in V} |\Gamma(v)| + Absorb(\xi, \mathbf{0}) \cdot 0 = \mathbf{E}_\xi[x_T] = \mathbf{E}_\xi[x_0] = \sum_{v \in V : \xi(v)=1} |\Gamma(v)|.$$

Thus we have Eq. (4).

As for (III), let $\mathbf{E}_\xi[T]$ be the expected agreement time necessary for the system to reach $\mathbf{0}/\mathbf{1}$ from ξ. Then [6, Theorem 7] states that $\mathbf{E}_\xi[T]$ satisfies the following difference equations:

$$\mathbf{E}_\xi[T] = \sum_{\eta \in \Xi} p(\xi, \eta) \mathbf{E}_\eta[T] + 1, \quad \mathbf{E}_\mathbf{0}[T] = \mathbf{E}_\mathbf{1}[T] = 0. \tag{7}$$

Thus $\mathbf{E}_\xi[T]$ is computable by solving a set of simultaneous linear equations with $2^n - 2$ variables, but obtaining its explicit form seems to be difficult.

Hassin and Peleg [4, Theorem 2] proposed an upper bound on the expected agreement time for n-polling, by using another Markov chain H in [4] with state space V.

Theorem 2 ([4, Theorem 2]) *If the Markov chain H is reversible, then the expected agreement time for δ_n-polling is $O(M \log n)$, where M is the maximal meeting time for two random walks on G.*

Note that Markov chain H in [4] is always reversible under the current setting. So $O(n^3 \log n)$ is an upper bound on the expected agreement time for δ_n-polling (assuming the definition of neighbors in [4]), because the meeting time is bounded by $O(n^3)$.

In this paper, we give an explicit upper bound on the expected agreement time for π-polling (assuming the definition neighbors in this paper). The bound depends on the initial global state ξ as well as G, so that we can obtain, for some initial global states, a better bound than a one depending only on G, such as the bound $O(n^3 \log n)$ in [4].

The rest of the paper is organized as follows: In Section 2, we first discuss general connected graphs and then complete graphs. The probabilistic polling games defined by δ_1- and δ_n-pollings on a complete graph are respectively known as Moran's and Wright-Fisher's models in population genetics [1]. Thus π-polling on the complete graph is a natural interpolation of the above two models. Concluding remarks are given in Section 3.

2 Upper Bounds on the Expected Agreement Time

2.1 General Connected Graphs

Let us recall an integer valued stochastic process x_t from Ξ-valued Markov chain X_t discussed in Theorem 1. Let

$$y_{t+1} = x_{t+1} - x_t, \tag{8}$$

which indicates the "efficiency" of polling at t.

Lemma 1 *The following statements hold for y_t.*

(i) $y_{t+1} = \sum_{v \in V : X_t(v)=0, X_{t+1}(v)=1} |\Gamma(v)| - \sum_{w \in V : X_{t+1}(w)=0, X_t(w)=1} |\Gamma(w)| \quad \mathbf{P}_\xi$-a.s,

(ii) $\mathbf{E}_\xi[y_t] = 0$ *for any t and* $\mathbf{E}_\xi[y_{t+1} | \mathcal{F}_t] = 0$ *a.s.*

Proof. Item (i) is obvious since the following equality holds:

$$\begin{aligned} &\{v \in V : X_t(v) = 1\} \cup \{v \in V : X_t(v) = 0, X_{t+1}(v) = 1\} \\ &\quad = \{v \in V : X_{t+1}(v) = 1\} \cup \{v \in V : X_t(v) = 1, X_{t+1}(v) = 0\}. \end{aligned}$$

Now we check Item (ii). Since $(x_t, \mathcal{F}_t)$ is martingale, $\mathbf{E}_\xi[x_t] = \mathbf{E}_\xi[x_0] = \sum_{v\in V:\xi(v)=1} |\Gamma(v)|$ for any t. Hence by definition $\mathbf{E}_\xi[y_t] = 0$ for any t. Because of Markov property of X_t and Eq. (6), we obtain that

$$\begin{aligned} \mathbf{E}_\xi[y_{t+1}|\mathcal{F}_t] &= \mathbf{E}_\xi[f(X_{t+1}) - f(X_t)|\mathcal{F}_t] = \mathbf{E}_\xi[f(X_{t+1})|\mathcal{F}_t] - f(X_t) \\ &= \sum_{\eta\in\Xi} p(X_t, \eta) f(\eta) - f(X_t) = f(X_t) - f(X_t) = 0 \quad a.s. \end{aligned}$$

So we have the desired results. ∎

We define the variance of y_t as $\sigma_t^2 = \mathbf{E}_\xi[(y_t)^2]$, which represents the "speed" of π-polling at time t. Clearly if σ_t^2 is small for any t then the expected agreement time is large. Remembering the definition of T as Eq. (3), put the non-trivial minimum of the variation

$$\sigma_{\min}^2 = \min_{\omega\in\Omega} \min_{1\le t\le T(\omega)} \sigma_t^2 > 0.$$

Note that $y_t = 0$ for $t \ge T+1$. For a technical matter, we define a new random variable as

$$\sigma_*^2(t) = \begin{cases} t\sigma_{\min}^2 & \text{if } t \le T, \\ T\sigma_{\min}^2 & \text{if } t > T. \end{cases}$$

Then we have the following lemma.

Lemma 2 *$(x_t^2 - \sigma_*^2(t), \mathcal{F}_t)$ is a submartingale.*

Proof. It is clear that $x_t^2 - \sigma_*^2(t)$ is $\mathcal{F}_t$-measurable and $\mathbf{E}_\xi[|f(X_t)^2 - \sigma_*^2(t)|] < \infty$ for any t because of the finiteness of the graph. Since x_t is $\mathcal{F}_t$-measurable, we have

$$\begin{aligned} &\mathbf{E}_\xi\left[x_{t+1}^2 - \sigma_*^2(t+1)\middle|\mathcal{F}_t\right] = \mathbf{E}_\xi\left[x_t^2 + 2x_t y_{t+1} + y_{t+1}^2 - \sigma_*^2(t+1)\middle|\mathcal{F}_t\right] \quad (9)\\ &= x_t^2 + 2x_t\mathbf{E}_\xi[y_{t+1}|\mathcal{F}_t] + \mathbf{E}_\xi[y_{t+1}^2|\mathcal{F}_t] - \mathbf{E}_\xi[\sigma_*^2(t+1)|\mathcal{F}_t] \\ &= x_t^2 + \mathbf{E}_\xi[y_{t+1}^2|\mathcal{F}_t] - \mathbf{E}_\xi[\sigma_*^2(t+1)|\mathcal{F}_t]. \end{aligned}$$

Now we consider the case of $t+1 \le T$. Then Eq. (9) is

$$x_t^2 + \mathbf{E}_\xi[y_{t+1}^2|\mathcal{F}_t] - (t+1)\sigma_{\min}^2 \ge x_t^2 - t\sigma_{\min}^2 = x_t^2 - \sigma_*^2(t).$$

On the other hand, we consider the case of $t+1 > T$. Then since $y_{t+1} = 0$, Eq. (9) is

$$x_t^2 - \mathbf{E}_\xi[T\sigma_{\min}^2|\mathcal{F}_t] = x_t^2 - T\sigma_{\min}^2 \ge x_t^2 - \sigma_*^2(t),$$

because T is $\mathcal{F}_t$-measurable. So we have the desired results. ∎

Considering the structure of any connected graph, we estimate $\sigma_{\min}^2$ as the following proposition.

Proposition 1 *Let d be the minimum degree of G, i.e., $d = \min_{v \in V} |\Gamma(v)| - 1$. Then*

$$\sigma^2_{\min} \geq \frac{d}{n(n-1)} \sum_{k=1}^{n} \pi_k k(2n-k-1). \tag{10}$$

Moreover if the polling is uniform one, that is, $\pi_k = \binom{n}{k}/(2^n - 1)$ for $k = 1, \cdots, n$ then

$$\sigma^2_{\min} \geq \frac{3d}{4 - 2^{-n+2}}. \tag{11}$$

Proof. By Item (i) of Lemma 1,

$$\sigma^2_{\min} = \min_{\xi \in \hat{\Xi}} \mathbf{E}_\xi \left[\left(\sum_{\xi(v)=1, X_1(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, X_1(w)=1} |\Gamma(w)| \right)^2 \right],$$

where $\hat{\Xi} = \Xi \setminus \{\mathbf{0}, \mathbf{1}\}$. By the definition of transition probability of Eq. (2),

$$\begin{aligned}
\sigma^2_{\min} &= \min_{\xi \in \hat{\Xi}} \left\{ \sum_{k=1}^{n} \frac{\pi_k}{\binom{n}{k}} \sum_{A \subseteq V} \left(\sum_{\xi(v)=1, \xi^A(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^A(w)=1} |\Gamma(w)| \right)^2 \right. \\
&\qquad \left. \sum_{U : A \subseteq U \in Sub_k(V)} \prod_{u \in U} \frac{N_\xi(u, \xi^A(u))}{|\Gamma(u)|} \right\} \\
&= \min_{\xi \in \hat{\Xi}} \left\{ \sum_{k=1}^{n} \frac{\pi_k}{\binom{n}{k}} \sum_{U \in Sub_k(V)} \sum_{A \subseteq U} \left(\sum_{\xi(v)=1, \xi^A(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^A(w)=1} |\Gamma(w)| \right)^2 \right. \\
&\qquad \left. \prod_{u \in U} \frac{N_\xi(u, \xi^A(u))}{|\Gamma(u)|} \right\}.
\end{aligned}$$

For a fixed ξ, let

$$V' = V'_\xi = \{v \in V : \text{there exists } w \in \Gamma(v) \text{ s.t. } \xi(v) \neq \xi(w)\}. \tag{12}$$

Then since $\xi \in \hat{\Xi}$, we have

$$|V'| \geq 2, \quad N_\xi(v,0), N_\xi(v,1) \geq 1, \quad N_\xi(v,0) + N_\xi(v,1) = |\Gamma(v)| \geq d+1, \quad v \in V'. \tag{13}$$

Thereby we deduce that

$$\begin{aligned}
\sigma^2_{\min} &= \min_{\xi \in \hat{\Xi}} \left\{ \sum_{k=1}^{n} \frac{\pi_k}{\binom{n}{k}} \sum_{U \in Sub_k(V)} \sum_{U \cap V' \neq \emptyset} \sum_{A \subseteq U} \right. \\
&\qquad \left. \left(\sum_{\xi(v)=1, \xi^A(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^A(w)=1} |\Gamma(w)| \right)^2 \prod_{u \in U} \frac{N_\xi(u, \xi^A(u))}{|\Gamma(u)|} \right\}.
\end{aligned}$$

For each $U \in Sub_k(V)$ satisfying $U \cap V' \neq \emptyset$, we put

$$I_\xi(U) = \sum_{A \subseteq U} \left(\sum_{\xi(v)=1, \xi^A(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^A(w)=1} |\Gamma(w)| \right)^2 \prod_{u \in U} \frac{N_\xi(u, \xi^A(u))}{|\Gamma(u)|}.$$

Letting any element $v_* \in U \cap V'$, we have that for $A = A_1 \cup A_2$

$$\begin{aligned} I_\xi(U) &= \sum_{A_1 \subseteq U \setminus \{v_*\}} \prod_{u \in U \setminus \{v_*\}} \frac{N_\xi(u, \xi^{A_1}(u))}{|\Gamma(u)|} \sum_{A_2 \subseteq \{v_*\}} \frac{N_\xi(v_*, \xi^{A_2}(v_*))}{|\Gamma(v_*)|} \\ &\quad \times \left(\sum_{\xi(v)=1, \xi^{A_1 \cup A_2}(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^{A_1 \cup A_2}(w)=1} |\Gamma(w)| \right)^2. \end{aligned}$$

Without loss of generality, assume $\xi(v_*) = 1$. Then since A_2 is $\{v_*\}$ or $\emptyset$, we have the estimation of the part after the second summation in $I_\xi(U)$ as

$$\begin{aligned} &\sum_{A_2 \subseteq \{v_*\}} \frac{N_\xi(v_*, \xi^{A_2}(v_*))}{|\Gamma(v_*)|} \left(\sum_{\xi(v)=1, \xi^{A_1 \cup A_2}(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^{A_1 \cup A_2}(w)=1} |\Gamma(w)| \right)^2 \quad (14) \\ &= \frac{N_\xi(v_*, 0)}{|\Gamma(v_*)|} \left(\sum_{\xi(v)=1, \xi^{A_1}(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^{A_1}(w)=1} |\Gamma(w)| + |\Gamma(v_*)| \right)^2 \\ &+ \frac{N_\xi(v_*, 1)}{|\Gamma(v_*)|} \left(\sum_{\xi(v)=1, \xi^{A_1}(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^{A_1}(w)=1} |\Gamma(w)| \right)^2. \end{aligned}$$

In general, for any real number z and $r \geq 2, 1 \leq w \leq r$,

$$\frac{w}{r}(z+r)^2 + \left(1 - \frac{w}{r}\right) z^2 = z^2 + 2wr + wr = (z+w)^2 + w(r-w) \geq w(r-w).$$

By Eq. (1), letting

$$z = \sum_{\xi(v)=1, \xi^{A_1}(v)=0} |\Gamma(v)| - \sum_{\xi(w)=0, \xi^{A_1}(w)=1} |\Gamma(w)|, \quad r = |\Gamma(v_*)|, w = N_\xi(v_*, 0),$$

we see that Eq. (14) is greater than or equal to $N_\xi(v_*, 0) N_\xi(v_*, 1)$. Therefore

$$\begin{aligned} I_\xi(U) &\geq N_\xi(v_*, 0) N_\xi(v_*, 1) \left(\sum_{A_1 \subseteq U \setminus \{v_*\}} \prod_{v \in U \setminus \{v_*\}} \frac{N_\xi(u, \xi^{A_1}(u))}{|\Gamma(u)|} \right) \\ &= N_\xi(v_*, 0) N_\xi(v_*, 1) \geq 1(d + 1 - 1) = d. \end{aligned}$$

Note that the above inequalities hold, by $v_* \in V'$ and Eq. (13). Hence

$$\sigma^2_{\min} \geq \min_{\xi \in \hat{\Xi}} \left\{ \sum_{k=1}^n \frac{\pi_k}{\binom{n}{k}} \sum_{U \in Sub_k(V)} \sum_{U \cap V' \neq \emptyset} d \right\} = d \sum_{k=1}^n \frac{\pi_k}{\binom{n}{k}} \min_{\xi \in \hat{\Xi}} |\{U \in Sub_k(V) : U \cap V' \neq \emptyset\}|$$

$$
\begin{aligned}
&\geq \quad d\sum_{k=1}^{n}\frac{\pi_k}{\binom{n}{k}}|\{U\in Sub_k(V): U\cap\{v_1',v_2'\}\neq\emptyset\}| = d\sum_{k=1}^{n}\frac{\pi_k}{\binom{n}{k}}\left\{\binom{n}{k}-\binom{n-2}{k}\right\}\\
&= \quad \frac{d}{n(n-1)}\sum_{k=1}^{n}\pi_k k(2n-k-1).
\end{aligned}
$$

So we have Eq. (10). It is clear that Eq. (11). ∎

By Lemma 1 and Proposition 1, we have the following theorem:

Theorem 3 *Let n and d be the order and the minimum degree of G, respectively, and let $\mathbf{E}_\xi[T]$ be the expected agreement time of the probabilistic π-polling game on G with an initial global state ξ. Then for any $1 \leq k \leq n$,*

$$
\mathbf{E}_\xi[T] \leq \frac{n(n-1)}{\sum_{k=1}^{n}\pi_k k(2n-k-1)d}\left(\sum_{v\in V:\xi(v)=0}|\Gamma(v)|\right)\left(\sum_{w\in V:\xi(w)=1}|\Gamma(w)|\right). \tag{15}
$$

Moreover if the polling is uniform one, that is, $\pi_k = \binom{n}{k}/(2^n-1)$ for $k = 1,\cdots,n$ then

$$
\mathbf{E}_\xi[T] \leq \frac{4-2^{-n+2}}{3d}\left(\sum_{v\in V:\xi(v)=0}|\Gamma(v)|\right)\left(\sum_{w\in V:\xi(w)=1}|\Gamma(w)|\right).
$$

Proof. By using Eq. (4) for initial state $\xi \in \hat{\Xi}$, we obtain the expectation of x_t^2 for $t = 0$ and $T = \inf\{t \in \mathbf{N} : X_t \in \{\mathbf{0},\mathbf{1}\}\}$ respectively:

$$
\mathbf{E}_\xi[x_T^2] = \left(\sum_{v\in V}|\Gamma(v)|\right)\left(\sum_{w\in V:\xi(w)=1}|\Gamma(w)|\right), \quad \mathbf{E}_\xi[x_0^2] = \left(\sum_{v\in V:\xi(v)=1}|\Gamma(v)|\right)^2. \tag{16}
$$

By Lemma 1, we hence apply the optional stopping theorem [5, pp 69, Remark] to $x_t^2 - \sigma_*^2(t)$ to have

$$
\mathbf{E}_\xi[x_T^2 - T\sigma_{\min}^2] \geq \mathbf{E}_\xi[x_0^2].
$$

By using Eq. (16) we have

$$
\mathbf{E}_\xi[T] \leq \frac{\mathbf{E}_\xi[x_T^2]-\mathbf{E}_\xi[x_0^2]}{\sigma_{\min}^2} = \frac{1}{\sigma_{\min}^2}\left(\sum_{v\in V:\xi(v)=0}|\Gamma(v)|\right)\left(\sum_{w\in V:\xi(w)=1}|\Gamma(w)|\right). \tag{17}
$$

By virtue of Proposition 1, we finally obtain Eq. (15). ∎

As we mentioned, Hassin and Peleg [4] adopt, for δ_n-polling and any vertex v, the neighborhood $\Gamma(v)$ that does not include v itself. For their setting, that is for non-bipartite graphs, we can obtain a similar result in analogy with Proposition 1 and Theorem 3:

$$\mathbf{E}_\xi[T] \le \frac{1}{\min_{v\in V} A(v)} \left(\sum_{v\in V:\xi(v)=0} d(v) \right) \left(\sum_{v\in V:\xi(v)=1} d(v) \right), \tag{18}$$

where $d(v)$ and $A(v)$ are the degree of v and

$$A(v) = |\{w \in \Gamma(v) \setminus \{v\} : \xi(w) = 1\}||\{u \in \Gamma(v) \setminus \{v\} : \xi(u) = 0\}|$$

respectively. Since the order of right hand side of Eq. (18) is $O(n^4)$, this bound is weaker than Theorem 2. However for some subclasses of graphs, we can obtain better bounds for δ_n-polling.

Corollary 1 **(dense)** *If* $d = \Theta(n)$ *then*

$$\max_\xi \mathbf{E}_\xi[T] = O(n^3), \tag{19}$$

where d *is the minimum degree.*

(non-dense) *If* $D = O(1)$ *then* $\max_\xi \mathbf{E}_\xi[T] = O(n^2)$*, where* D *is the maximal degree of the graph, that is,* $D = \max_{v\in V} d(v) = \max_{v\in V} |\Gamma(v)| - 1$.

2.2 Complete Graphs

When G is a complete graph, a global state is characterized by the number of 1's (i.e., vertices with value 1) in it. We therefore use the global state space $S = \{0, 1, \cdots, n\}$ instead of $\Xi = \{0, 1\}^V$. By the definition of π-polling, an S-valued Markov chain, Z_t, is associated. Let $p(i, j)$ be the transition probability from global state $i \in S$ to $j \in S$, that is

$$p(i, j) = \mathbf{P}\{Z_{t+1} = j | Z_t = i\}. \tag{20}$$

Lemma 3 $p(i, j) = 1$*, if* $(i, j) \in \{(0, 0), (n, n)\}$,

$$p(i,j) = \sum_{k=1}^{n} \frac{\pi_k}{\binom{n}{k}} \sum_{l=\max\{0,k+i-n\}}^{\min\{k,i\}} \binom{n-i}{k-l} \binom{i}{l} \binom{k}{j-i+l} \left(\frac{i}{n}\right)^{j-i+l} \left(1-\frac{i}{n}\right)^{k-j+i-l}, \tag{21}$$

if $\Big\{(i, j) : 1 \le i \le n-1,\ i - \min\{k, i\} \le j \le k + i - \max\{0, k + i - n\}$ *for some* k *with* $\pi_k > 0\Big\}$*, and* $p(i, j) = 0$ *otherwise.*

Proof. By definition $p(0, 0) = p(1, 1) = 1$ holds. The distribution concerning π-polling is hypergeometrical: Let us randomly select k vertices from V with probability π_k constructed by i vertices with value 1 and $n - i$ vertices with value 0. Then the probability that exactly l vertices in the selected k vertices have value 1 is

$$\binom{n-i}{k-l}\binom{i}{l}\binom{n}{k}^{-1}, \tag{22}$$

where $\max\{0, k+i-n\} \leq l \leq \min\{k, i\}$. Let m be the number of vertices with value 1 after updating the selected k vertices. The probability of obtaining m vertices with value 1 is calculated by the binomial distribution with parameter i/n, that is,

$$\binom{k}{m}\left(\frac{i}{n}\right)^m\left(1-\frac{i}{n}\right)^{k-m}, \quad 0 \leq m \leq k, \tag{23}$$

because of the completeness of the graph. Moreover the changing number of vertices with value 1 is $m-l$ for the update. On the other hand, assuming the transition from i to j, we have that $m = j - i + l$. Since the events of selecting and updating are independent, the transition probability is the sum of the product of Eqs. (22),(23) for possible terms. Hence we have Eq. (21). Clearly the probability is 0, otherwise. ∎

By Lemma 3, the transition probabilities for δ_1- and δ_n-pollings are

$$\begin{aligned} p(i,j) &= \begin{cases} (i/n)^2 + (1-i/n)^2, & \text{for } i = j, i \in S, \\ i(1-i/n)/n, & \text{for } j = i+1, i-1, i = 1, \cdots, n-1, \\ 0, & \text{otherwise,} \end{cases} \\ p(i,j) &= \binom{n}{j}\left(\frac{i}{n}\right)^j\left(\frac{n-i}{n}\right)^{n-j} \qquad i, j \in S. \end{aligned}$$

These probabilities are known as Moran's and Wright-Fisher's models in population genetics, respectively (E.g., see [1] and [7, Examples 5.1.3 and 5.1.4]), and their expected agreement times are well-known [7, pp 178]:

(δ_1-polling) $\mathbf{E}_{np}[T] = -n^2\{p\log p + (1-p)\log(1-p)\}(1+o(1)) = \Theta(n^2)$.

(δ_n-polling) $\mathbf{E}_{np}[T] = -2n\{p\log p + (1-p)\log(1-p)\}(1+o(1)) = \Theta(n)$,

where $p \in (0,1)$ is the ratio to n of the number of 1's in the initial state. The following theorem that treats π-polling is an "interpolation" of them.

Theorem 4 *Let T^{comp} be the agreement time for π-polling on n-complete graphs with an initial state satisfying that the proportional ratio of the number of 1's in the state to n is $p \in (0,1)$. Then*

$$\mathbf{E}_{np}[T^{comp}] = -2\left\{p\log p + (1-p)\log(1-p)\right\}\frac{n^2(n-1)(1+o(1))}{\sum_{k=1}^{n}\pi_k k(2n-k-1)}. \tag{24}$$

Moreover if the polling is uniform one, that is, $\pi_k = \binom{n}{k}/(2^n-1)$ for $k = 1, \cdots, n$ then

$$\mathbf{E}_{np}[T^{comp}] = -\frac{4}{3}\left\{p\log p + (1-p)\log(1-p)\right\}(1+o(1))n. \tag{25}$$

Some properties are needed for the proof of Theorem 4. Recalling Eq. (20), let $W_{t+1} = Z_{t+1} - Z_t$. Then we also use the following facts:

Lemma 4 *For any $t = 0, 1, \cdots$ and $k = 1, \cdots, n$, $\mathbf{E}[W_t] = 0$ and*

$$\mathbf{E}[(W_t)^2 | Z_t = i] = \sum_{k=1}^{n} \frac{\pi_k k(2n-k-1)}{n-1} \frac{i}{n} \left(1 - \frac{i}{n}\right). \tag{26}$$

Moreover if $\pi_k = \binom{n}{k}/(2^n - 1)$ for $k = 1, \cdots, n$ then

$$\mathbf{E}[(W_t)^2 | Z_t = i] = \frac{3i}{4 - 2^{-n+2}} \left(1 - \frac{i}{n}\right). \tag{27}$$

Proof. It is clear that $\mathbf{E}[W_t] = 0$, since $\mathbf{E}[Z_t]$ is independent of t. It is well-known that the first and second moments of binomial and hypergeometrical distributions is the following:

$$\begin{aligned}
&\sum_{j=0}^{k} j \binom{k}{j} p^j (1-p)^{k-j} = kp, \\
&\sum_{j=0}^{k} j^2 \binom{k}{j} p^j (1-p)^{k-j} = kp\{(k-1)p + 1\}, &(28)\\
&\frac{1}{\binom{n}{k}} \sum_{l=\max\{0,k+i-n\}}^{\min\{k,i\}} l \binom{n-i}{k-l} \binom{i}{l} = \frac{ki}{n}, &(29)\\
&\frac{1}{\binom{n}{k}} \sum_{l=\max\{0,k+i-n\}}^{\min\{k,i\}} l^2 \binom{n-i}{k-l} \binom{i}{l} = \frac{ki}{n} \left(1 + \frac{(k-1)(i-1)}{n-1}\right). &(30)
\end{aligned}$$

Using Lemma 3, for $i = 1, \cdots, n-1$

$$\mathbf{E}[(W_t)^2 | Z_t = i] + i^2 = \mathbf{E}[(Z_{t+1})^2 | Z_t = i] = \sum_{j=0}^{n} j^2 p(i,j)$$

$$= \sum_{k=1}^{n} \frac{\pi_k}{\binom{n}{k}} \sum_{l=\max\{0,k+i-n\}}^{\min\{k,i\}} \binom{n-i}{k-l} \binom{i}{l} \sum_{j=i-l}^{k+i-l} j^2 \binom{k}{j-i+l} \left(\frac{i}{n}\right)^{j-i+l} \left(1 - \frac{i}{n}\right)^{k-j+i-l}.$$

Letting $A = \sum_{j=i-l}^{k+i-l} j^2 \binom{k}{j-i+l} \left(\frac{i}{n}\right)^{j-i+l} \left(1 - \frac{i}{n}\right)^{k-j+i-l}$, by Eq (28),

$$\begin{aligned}
A &= \sum_{j=0}^{k} (j+i-l)^2 \binom{k}{j} \left(\frac{i}{n}\right)^j \left(1 - \frac{i}{n}\right)^{k-j} \\
&= k\left(\frac{i}{n}\right) \left\{(k-1)\frac{i}{n} + 1\right\} + 2(i-l)k\left(\frac{i}{n}\right) + (i-l)^2 \\
&= l^2 - 2\left(i + \frac{ki}{n}\right) l + \frac{ki}{n} \left\{(k-1)\frac{i}{n} + 1 + 2i\right\} + i^2.
\end{aligned}$$

By Eqs (29) and (30),

$$\mathbf{E}[(W_t)^2|Z_t = i] = \sum_{k=1}^{n} \pi_k \left\{ \frac{ki}{n}\left(1 + \frac{(i-1)(k-1)}{n-1}\right) + \frac{ki}{n}\left(-2i - \frac{2ki}{n}\right) \right.$$
$$\left. + \frac{ki}{n}\left((k-1)\frac{i}{n} + 1 + 2i\right)\right\} = \sum_{k=1}^{n} \frac{\pi_k k(2n-k-1)}{n-1}\frac{i}{n}\left(1 - \frac{i}{n}\right).$$

Moreover Eq. (27) is straightforward. ∎

The proof of Theorem 4 is essentially due to [1, pp 75]. The method is based on an approximation of the solution of Eq. (7) for the transition probability in Lemma 3. Now we follow [1, pp 75] as the proof.

Proof of Theorem 4. For sufficient large n, put $i/n = p > 0$. Assuming that $Z_t = i$ and letting $W = W_t/n$, we have by Lemma 4,

$$\mathbf{E}[W] = 0, \quad \mathbf{E}[W^2] = \sum_{k=1}^{n} \frac{\pi_k k(2n-k-1)}{n^2(n-1)} p(1-p).$$

For large n we can regard $T_k(i)$ as $T(p)$ which is twice differentiable. Then Eq (7) is that

$$T(p) = \mathbf{E}[T(p+W)] + 1, \quad 0 < p < 1, \quad T(0) = T(1) = 0.$$

Using the Taylor expansion of $T(p+W)$,

$$\begin{aligned} T(p) &= \mathbf{E}[T(p+W)] + 1 = \mathbf{E}[T(p) + T'(p)W + T''(p)W^2/2 + o(W^2))] + 1 \\ &= T(p) + T'(p)\mathbf{E}[W] + T''(p)\mathbf{E}[W^2]/2 + o(\mathbf{E}[W^2]) + 1. \end{aligned}$$

So we obtain

$$T''(p) = -2\frac{n^2(n-1)(1+o(1))}{\sum_{k=1}^{n} \pi_k k(2n-k-1)} \frac{1}{p(1-p)}.$$

Now by the boundary conditions $T(0) = T(1) = 0$, we have that

$$T(p) = 2(-p\log p - (1-p)\log(1-p))\frac{n^2(n-1)(1+o(1))}{\sum_{k=1}^{n} \pi_k k(2n-k-1)}.$$

Eq. (25) is straightforward. ∎

3 Conclusion

In this paper, we gave some upper bounds on the expected agreement time of the probabilistic polling game defined by π-polling.

If the graph is uniformly dense or non-dense then the order of expected agreement time seems not to be so large by Corollary 1. Therefore we conjecture that a pair of a graph and an initial global state for it that achieve the worst expected agreement time is given in Fig. 1. The graph there is called *barbell* graph, which consists of two copies of $n/3$-cliques connected by a path graph of length $n/3$. In the initial global state, the vertices with value 0 (1) are colored white (black). Hence there is only one edge connecting vertices with different values.

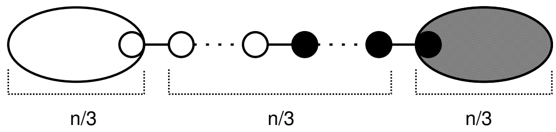

Figure 1: A barbell graph: two copies of $n/3$-cliques are connected by a path graph of length $n/3$. Vertices with value 0 (1) are colored white (black).

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