Buffon's needle on a square lattice

The Buffon needle problem (1777) is the following. We randomly drop a needle of unit length on a plane with a grid of parallel lines $y = n$ ($n = 0, \pm 1, \pm 2, \ldots$). What is the probability that it will intersect some line? By setting a probability space explicitly, we get the answer is $2/\pi$ (see [1, p.100]). The purpose of this problem is to get an approximation of π. Let X be a number of intersections of the needle with the grid. Since X is distributed with $\Pr(X = 1) = 1 - \Pr(X = 0) = 2/\pi$, the expectation and the variance are respectively

$$\mathbb{E}(X) = \frac{2}{\pi} \quad \text{and} \quad \mathrm{var}(X) = \frac{2}{\pi} - \frac{4}{\pi^2}.$$

A variant of this is given in [2, Problem 4.5.3, p.31]. Instead of the needle, we drop a cross formed by welding together two unit needles perpendicularly at their midpoints, which is called *Buffon's cross*. Let Y be a number of intersections of the cross with the grid. Along the solution [2, Solution 4.5.3, p.192] we can obtain

$$\begin{cases} \Pr(Y = 2) = \frac{2(2-\sqrt{2})}{\pi}, \\ \Pr(Y = 1) = \frac{4(\sqrt{2}-1)}{\pi}, \\ \Pr(Y = 0) = 1 - \frac{2\sqrt{2}}{\pi}, \end{cases}$$

which yield

$$\mathbb{E}(Y/2) = \frac{2}{\pi} \quad \text{and} \quad \mathrm{var}(Y/2) = \frac{3-\sqrt{2}}{\pi} - \frac{4}{\pi^2}.$$

Note that $\mathbb{E}(Y) = 2\mathbb{E}(X)$ can be interpreted as linearity of the expectation, since the cross is constructed by two needles of unit length (see also Barbier's theorem [3, p.508]). When considering $Y/2$ and X as unbiased estimators for $2/\pi$, the estimator $Y/2$ is more efficient than X since $\mathrm{var}(Y/2) < \mathrm{var}(X)$.

In this note, we propose another efficient unbiased estimator. Let us construct a square lattice by considering two grids of parallel lines are superimposed: the first grid is $y = n$ ($n = 0, \pm 1, \pm 2, \ldots$), and the second is $x = n$ ($n = 0, \pm 1, \pm 2, \ldots$) which are perpendicular to those of the first set. Let Z be a number of intersections of a needle of unit length with the square lattice, and $Z/2$ is the estimator. This setting is due to [2, Problem 4.5.2, p.31] for $a = b = r = 1$, which is the problem of showing the probability that the needle intersects the lattice is $3/\pi$. It is called the *Laplace extension of Buffon's problem*, which is studied by [4] with discussions of variance for n throws of needles, and numerically and contemporarily studied by [5]. Although this problem is independent of Buffon's cross, we regard it as a *companion problem*.

Let $(\Omega, \Pr)$ be a probability space, which is a little simpler than [2, Solution 4.5.2, p.191], satisfying $\Omega = \{(x, y, \theta) : x, y \in [0, 1/2], \theta \in [0, \pi/2]\}$ and $\Pr(B) = |B|/|\Omega|$ for the volume measurable event $B \subset \Omega$, where $|\cdot|$ denotes the volume. Indeed, suppose that the midpoint of the needle randomly falls to a unit square with a random angle. Then, for $(x, y, \theta) \in \Omega$, x and y denote the nearest distances between the point and the first and the second grids respectively, and $\theta = \min\{\theta', \pi - \theta'\}$, where $0 \leq \theta' \leq \pi$ is the angle between the needle and the first grid. Since events for Z are described as

$$\begin{cases} \{Z = 2\} = \bigcup_{\theta \in \left[0, \frac{\pi}{2}\right]} \left\{(x, y, \theta) \in \Omega : 0 < x < \frac{\cos\theta}{2}, 0 < y < \frac{\sin\theta}{2}\right\}, \\ \{Z = 0\} = \bigcup_{\theta \in \left[0, \frac{\pi}{2}\right]} \left\{(x, y, \theta) \in \Omega : \frac{\cos\theta}{2} < x < \frac{1}{2}, \frac{\sin\theta}{2} < y < \frac{1}{2}\right\}, \\ \{Z = 1\} = \Omega \setminus (\{Z = 0\} \cup \{Z = 2\}), \end{cases}$$

it turns out that

$$\begin{cases} \Pr(Z = 2) = \frac{|\{Z=2\}|}{\pi/8} = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{2} \cdot \frac{\sin\theta}{2} d\theta = \frac{1}{\pi}, \\ \Pr(Z = 0) = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{\cos\theta}{2}\right)\left(\frac{1}{2} - \frac{\sin\theta}{2}\right) d\theta = 1 - \frac{3}{\pi}, \\ \Pr(Z = 1) = \frac{2}{\pi}. \end{cases}$$

Therefore $\mathbb{E}(Z) = \frac{4}{\pi}$ and $\mathrm{var}(Z) = \frac{6}{\pi} - \left(\frac{4}{\pi}\right)^2$ hold. From this it follows that

$$\mathbb{E}(Z/2) = \frac{2}{\pi} \quad \text{and} \quad \mathrm{var}(Z/2) = \frac{3}{2\pi} - \frac{4}{\pi^2}.$$

While estimators $X, Y/2$ and $Z/2$ are unbiased, $Z/2$ is the most efficient of them because of $\mathrm{var}(Z/2) < \mathrm{var}(Y/2) < \mathrm{var}(X)$.

Note that $\mathbb{E}(Z) = 2\mathbb{E}(X)$ can also be interpreted as linearity of the expectation, since there exist two grids. Moreover, for the unbiased estimators $Z/2$ and $Y/2$, the inequality $\mathrm{var}(Z/2) < \mathrm{var}(Y/2)$ is deduced from the inequality $\Pr(Z = 2) < \Pr(Y = 2)$ which means that the probability that the needle intersects twice is smaller than the probability that the cross intersects twice.

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