Approximations for the Feller games

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1. *Introduction*

Suppose that a coin is tossed repeatedly until the same number of Heads as Tails. We consider a game that a player gets X yen if the number of coin tosses is X. For example, if TH, HHTT and TTHTHH then $X = 2, X = 4$ and $X = 6$, respectively, where H denotes Heads and T denotes Tails. Table A below shows a result of 20 students in a classroom doing this experiment.

Student No.	1	2	3	4	5	6	7	8	9	10
X	2	6	20	6	6	2	2	1052	2	2
Student No.	11	12	13	14	15	16	17	18	19	20
X	34	22	2	2	32	6	48	2	6	2

Table A

It is reasonable that 9 students have $X = 2$, while $X = 1052$ of the 8th student seems too large. Is this unusual in the sense of statistics? Referring the *St. Petersburg game*, Treviño [1] reported several properties of X, and illustrated numerical results. However the calculations are a little troublesome even with computers, because they are exact values.

In this Note, after remarking them, we give some approximate results with easy calculations using several results of limit distributions. Moreover we study medians of both the sample mean and the maximum with respect to X.

2. *The Feller game vs the St. Petersburg game*

This game is written in Feller's textbook [2, Section X.1, page 246]. Hence Matsumoto and Nakata [3] called it *the Feller game*. The distribution of X is

$$\mathrm{P}(X = 2k) = \frac{2}{4^k k}\binom{2k-2}{k-1} = \frac{1}{2k-1}\binom{2k}{k}2^{-2k}. \tag{1}$$

The first equality is due to [1, Theorem 1], and the second one is given by [2, Equation (3.7), page 78] [3, Equation (1)], or [4, Exercise 3.10.1, page 83].

By the way, the St. Petersburg game, which was firstly published by Daniel Bernoulli (1738) in Saint Petersburg (Russia), is well-known as follows (see [3, Section X.4, page 251]). Consider that a coin is tossed repeatedly until it falls Heads. If it happens at the kth trial then the player gets 2^k yen. Letting Y be the payoff of the game, we have

$$\mathrm{P}(Y = 2^k) = 2^{-k} \quad \text{for } k = 1, 2, \dots \tag{2}$$

For these games it follows that $\mathrm{E}(X) = \mathrm{E}(Y) = \infty$. In fact,

$$\mathrm{E}(Y) = 2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right)^2 + \cdots = \infty.$$

Similarly, direct proofs of $\mathrm{E}(X) = \infty$ are given in [1, Theorem 2] and [3, Lemma 2]. Note that it can be interpreted as a simple random walk starting from the origin. Indeed, it follows from the fact the random walk returns to the origin with probability one, but the expected return steps are infinite.

It should be pointed out that asymptotic properties of the tail probabilities for X and Y are different. The Stirling formula, which is $n! \sim \sqrt{2\pi n}(n/e)^n$, yields

$$\mathrm{P}(X > x) \sim \sqrt{\frac{2}{\pi}} x^{-1/2}, \tag{3}$$

where $f(x) \sim g(x)$ denotes $\lim_{x\to\infty} f(x)/g(x) = 1$. Equation (3) follows from [3, Equation (23)]. By contrast, we have

$$\mathrm{P}(Y > x) = 2^{\{\log_2 x\}} x^{-1} \quad \text{for } x > 2, \tag{4}$$

whose validation is also given in [3, Equation (3)], where $\{x\}$ is the fractional part of $x > 0$. In other words, while it follows that $\lim_{x\to\infty} \sqrt{x}\mathrm{P}(X > x) = \sqrt{2/\pi}$, there does not exist a limit of $x\mathrm{P}(Y > x) = 2^{\{\log_2 x\}}$. Hence it is not so easy to investigate the limit distribution of Y compared to X. Details are discussed in [3].

3. *Feller games for n students*

3.1. *The number of students with more than m trials*

We proceed with the investigation of Treviño [1, page 38] that each of n students tosses a coin until they each have the same number of Heads as Tails. For $i = 1, \dots, n$ let X_i be the number of coin tosses of the ith student. Note that $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) with the common distribution (1). We define

$$\mathbb{I}_i = \mathbb{I}_i(m) = \begin{cases} 1, & \text{if } X_i \geq m, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad W_n = W_n(m) = \sum_{i=1}^{n} \mathbb{I}_i, \tag{5}$$

respectively. Then $\{\mathbb{I}_i\}$ are also i.i.d., and can be regarded as other coin tosses with success probability

$$p_m = \mathrm{E}(\mathbb{I}_1) = \mathrm{P}(\mathbb{I}_1 = 1) = \mathrm{P}(X \geq m) = \sum_{k=\lceil m/2 \rceil}^{\infty} \frac{1}{2k-1} \binom{2k}{k} 2^{-2k}, \qquad (6)$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to $x > 0$. Hence it turns out that W_n is binomially distributed with parameters n and p_m. A list of numeric probabilities

$$\mathrm{P}(W_n \geq 1) = 1 - (1 - p_m)^n \qquad (7)$$

for m and n is illustrated in [1, Table 1, page 39], which are the probabilities that at least one of the students tosses a coin m or more trials. For example, when $m = 1000$ and $n = 20$, we can see $\mathrm{P}(W_{20}(1000) \geq 1) = 0.4004$. Hence $X = 1052$ in Table A is not so outrageous.

Moreover a list of integers

$$a_m = \min \left\{ n \geq 1 \;\middle|\; \mathrm{P}(W_n(m) \geq 1) \geq \frac{1}{2} \right\} \qquad (8)$$

for m is also illustrated in [1, Table 2, page 39], which are the numbers of students needed to have a better than even chance that at least one of them will toss the coin at least m trials. For example, when $m = 1000$, we can see $a_{1000} = 28$. Hence if the number of students in the classroom is 28 or more, the probability that some of them toss 1000 or more times is at least 1/2.

3.2. *Poisson approximations for W_n*

Since p_m satisfies the approximation

$$p_m \sim \sqrt{\frac{2}{\pi}} m^{-1/2} \qquad (9)$$

when m is large enough because of (3), the event $\{X \geq m\}$ may be considered as a rare event in this case. In addition, for large $n = n(m)$ satisfying that $n/\sqrt{m}$ is near a positive constant, we have the following Poisson approximation.

$$\mathrm{P}(W_n = l) \sim e^{-np_m} \frac{(np_m)^l}{l!} \quad \text{for } l = 0, 1, \ldots. \qquad (10)$$

Remark: For *the Birthday problem*, we have good Poisson approximations (see [2, Example (b), page 105]). It follows from the facts that the number of trials, which are $\binom{n}{2}$ for n students, is large, and the success probability, which is 1/365, is small. Note that $\binom{n}{2}$ is large even if n is not so large.

Theorem 1: If n and m satisfy (9) and (10) then we have the following approximations.

1. The probabilities of (7) are approximated by

$$\mathrm{P}(W_n \geq 1) \sim 1 - e^{-\sqrt{\frac{2}{\pi}}\frac{n}{\sqrt{m}}}. \tag{11}$$

2. The integers of (8) are approximated by

$$a_m \sim \left\lceil (\log 2)\sqrt{\frac{\pi m}{2}} \right\rceil. \tag{12}$$

Proof:

1. Equations (10) and (9) yield

$$\mathrm{P}(W_n \geq 1) = 1 - \mathrm{P}(W_n = 0) \overset{(10)}{\sim} 1 - e^{-np_m} \overset{(9)}{\sim} 1 - e^{-\sqrt{\frac{2}{\pi}}\frac{n}{\sqrt{m}}}.$$

2. We solve the inequality $\mathrm{P}(W_n \geq 1) \geq 1/2$ for large n. Since $1 - e^{-np_m} \geq 1/2$, we have

$$a_m \sim \left\lceil \frac{\log 2}{p_m} \right\rceil \overset{(9)}{\sim} \left\lceil (\log 2)\sqrt{\frac{\pi m}{2}} \right\rceil.$$

Note that if $\frac{n}{\sqrt{m}}$ is positive small then $\mathrm{P}(W_n \geq 1) \sim \sqrt{\frac{2}{\pi}}\frac{n}{\sqrt{m}}$. The calculation of both this and (12) would be possible with a simple calculator. Decimal outputs of (11) for m and n are near values of [1, Table 1, page 39]. Similarly, integer outputs of (12) for m is equivalent to [1, Table 2, page 39].

3.3. *The median of the sample mean*

For i.i.d. random variables $X_1, \ldots, X_n$ which are the trial numbers of coin tosses of n students, we investigate the sample mean $\overline{X} = S_n/n$, where $S_n = \sum_{i=1}^{n} X_i$. Since $\mathrm{E}(X_1) = \infty$, we have $\mathrm{E}(\overline{X}) = \infty$. Therefore it is difficult even to roughly estimate $\overline{X}$. Hence we examine the median of $\overline{X}$ instead of $\mathrm{E}(\overline{X})$. In [2, Equation (1.7), page 246], Feller calculated

$$\mathrm{P}\left(\frac{S_n}{n^2} < x\right) \sim 2\left(1 - \Phi\left(\frac{1}{\sqrt{x}}\right)\right) \quad \text{for } x > 0, \tag{13}$$

where $\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}/\sqrt{2\pi}dt$. The right hand side of (13) is called *the one-sided stable distribution with index* $1/2$ or *the Lévy distribution*, whose probability density function is $e^{-1/(2x)}/(\sqrt{2\pi}x^{3/2})$ for $x > 0$ (see [3, Theorem 1 and references there in]). Considering (13), we solve the equation $2\left(1 - \Phi\left(\frac{1}{\sqrt{x}}\right)\right) = \frac{1}{2}$ with $x > 0$. The unique solution is $x = \left(1/\Phi^{-1}(0.75)\right)^2 \sim 2.198$. Hence it turns out that

$$\mathrm{P}\left(\overline{X} < 2.198n\right) = \mathrm{P}\left(S_n < 2.198n^2\right) \sim \frac{1}{2}.$$

Thus we have the following theorem.

Theorem 2: The median of $\overline{X}$ is near $2.198n$.

3.4. *The median of the maximum*

We also consider the maximum M_n for $X_1, \ldots, X_n$, where $M_n = \max\{X_1, \ldots, X_n\}$. Since $\mathrm{E}(M_n) = \infty$, we study the median of M_n corresponding to $\overline{X}$. It is known that

$$\mathrm{P}\left(\frac{M_n}{n^2} < x\right) \sim \exp\left(-x^{-\frac{1}{2}}\right) \quad \text{for } x > 0, \tag{14}$$

which is called *the Fréchet distribution* with index 1/2 (see [3, Equation (14)]). Considering (14), we solve the equation $\exp\left(-x^{-\frac{1}{2}}\right) = \frac{1}{2}$ with $x > 0$. The unique solution is $x = (\log 2)^{-2} \sim 2.081$. Hence it turns out that $\mathrm{P}\left(M_n < 2.081n^2\right) \sim \frac{1}{2}$. Thus we have the following theorem.

Theorem 3: The median of M_n is near $2.081n^2$.

4. *Concluding remarks*

Be careful when letting students play the Feller games in the classroom. Indeed, many students will finish in a small number of trials, but a few students may not finish during class hours.

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